Time Dependence of Expectation Values

The average or expected value of the measurement of the classical variable, a, associated with the operator $\hat{A}$ is $\langle\psi|\hat{A}|\psi\rangle \equiv \langle a\rangle$

<u>Proof</u> Since $\hat{A}$ is Hermitian its eigenvectors span the state space and may be converted (Gram-Schmidt) into the orthonormal basis for the space $\{|a_n\rangle\}$.

Since $\{|a_n\rangle\}$ spans,

$$|\psi\rangle = \sum_n c_n |a_n\rangle$$

therefore,

$$\langle a\rangle = \langle\psi|\hat{A}|\psi\rangle$$
$$= \left(\sum_i c_i^* \langle a_n|\right) \hat{A} \left(\sum_j c_j |a_j\rangle\right)$$

where $\langle\psi| = \sum c_i^* \langle a_i|$

so $\left(\sum_i c_i^* \langle a_i|\right)\left(\sum_j c_j a_j |a_j\rangle\right)$

where I have used

$$\hat{A}|a_n\rangle = a_n |a_n\rangle$$

or

$$\langle a\rangle = \sum_{ij} c_i^* c_j a_j \langle a_i | a_j\rangle$$

$$= \sum_{ij} c_i^* c_j a_j \delta_{ij} = \sum_i c_i^* c_i a_i$$

By the postulates, the probability to observe a_n is $P(a_n) = c_n^* c_n$

so $\langle a\rangle = \sum a_n P(a_n)$ = average of a_n

Q.E.D

The ~~wave~~ state vector $|\psi\rangle$ evolves in time according to the Schrodinger eqn

$$i\hbar \frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle$$

so we can write $|\psi(t)\rangle$ and compute the expectation value as a function of time.

$$\langle a\rangle(t) = \langle \psi(t)|\hat{A}|\psi(t)\rangle$$

The rate of change of the expectation value is

$$\frac{d\langle a\rangle}{dt} = \left(\frac{d\langle\psi(t)|}{dt}\right)\hat{A}\,|\psi(t)\rangle$$

$$+ \langle\psi(t)|\,\frac{\partial\hat{A}}{\partial t}\,|\psi(t)\rangle$$

$$+ \langle\psi(t)|\hat{A}\,\frac{d}{dt}\,|\psi(t)\rangle$$

I'll let the t dependence be implied.

Solving the SE yields,

$$\frac{d|\psi\rangle}{dt} = \frac{-i}{\hbar}\hat{H}|\psi\rangle$$

and taking the adjoint $(\hat{H} = \hat{H}^{\dagger})$

$$\frac{d}{dt}\langle\psi| = \frac{i}{\hbar}\langle\psi|\hat{H}$$

Substitute

$$\frac{d\langle a\rangle}{dt} = \frac{i}{\hbar}\langle\psi|\hat{H}\hat{A}|\psi\rangle$$

$$+ \langle\psi|\frac{\partial\hat{A}}{\partial t}|\psi\rangle$$

$$-\frac{i}{\hbar}\langle\psi|\hat{A}\hat{H}|\psi\rangle$$

$$\frac{d\langle o\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{A}]|\psi\rangle + \left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle$$

$\Rightarrow$ This contains all of classical mechanics

$\Rightarrow$ Ehrenfest's Thm

$\Rightarrow$ If $\hat{H} = \hat{A}$

$$\frac{d\langle E\rangle}{dt} = 0 \Rightarrow \text{Energy is conserved}$$

<u>Ex</u> Consider a particle moving under a constant force $\vec{F} = f\hat{x}$

$$\vec{F} = -\nabla V$$

$$V = -fx$$

<u>Hamiltonian</u>

$$\hat{H} = \frac{\hat{P}^2}{2m} + V = \frac{\hat{P}^2}{2m} - f\hat{x}$$

$$\frac{\partial\hat{H}}{\partial t} = 0$$

$$\frac{d}{dt}\langle x\rangle = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{x}]|\psi\rangle$$

$$\frac{d}{dt}\langle p\rangle = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{p}]|\psi\rangle$$

<u>Commutators</u>

$$[\hat{H},\hat{x}] = \left[\frac{\hat{p}^2}{2m} - f\hat{x}\,,\,\hat{x}\right]$$

$$= \frac{1}{2m}[\hat{p}^2,\hat{x}] - f\underbrace{[\hat{x},\hat{x}]}_{0}$$

$$[\hat{p}^2,\hat{x}] = \hat{p}\hat{p}\hat{x} - \hat{x}\hat{p}\hat{p}$$

but $[\hat{x},\hat{p}] = i\hbar = \hat{x}\hat{p} - \hat{p}\hat{x}$

$$\hat{x}\hat{p} = i\hbar + \hat{p}\hat{x}$$

$$[\hat{p}^2,\hat{x}] = \hat{p}\hat{p}\hat{x} - (i\hbar + \hat{p}\hat{x})\hat{p}$$

$$= \hat{p}\hat{p}\hat{x} - \hat{p}\hat{x}\hat{p} - i\hbar\hat{p}$$

$$= \hat{p}[\hat{p},\hat{x}] - i\hbar\hat{p} = -2i\hbar\hat{p}$$

$$[\hat{H}, \hat{X}] = \frac{-2i\hbar}{2m}\hat{P} = \frac{-i\hbar}{m}\hat{P}$$

$$\frac{d}{dt}\langle x\rangle = \frac{i}{\hbar}\langle\psi| -\frac{i\hbar}{m}\hat{P}|\psi\rangle$$

$$= \frac{1}{m}\langle\psi|\hat{P}|\psi\rangle = \frac{1}{m}\langle p\rangle \checkmark$$

Likewise,

$$[\hat{H}, \hat{P}] = \left[\frac{\hat{P}^2}{2m} - f\hat{X}, \hat{P}\right]$$

$$= -f[\hat{X}, \hat{P}] = -i\hbar f$$

$$\frac{d\langle p\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[\hat{H}, \hat{P}]|\psi\rangle$$

$$= \frac{i}{\hbar}\langle\psi| -i\hbar f|\psi\rangle = f\langle\psi|\psi\rangle$$

$= f$ <u>Newton II</u>