Time Independent Schrodinger Eqn

SE $$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi$$

Try separation of variables

$$\psi(x,t) = \phi(x)\,T(t)$$

Substitute

$$i\hbar\, \phi(x) \frac{\partial T}{\partial t} = -\frac{\hbar^2}{2m} T(t) \frac{\partial^2 \phi}{\partial x^2} + V\phi T$$

Divide by ϕT

$$i\hbar \frac{1}{T} \frac{\partial T}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} + V$$

The two sides are functions of different variables. To be equal for all x, t, both sides must equal a separation constant E.

$$i\hbar \frac{1}{T}\frac{dT}{dt} = E = -\frac{\hbar^2}{2m}\frac{1}{\phi}\frac{d^2\phi}{dx^2} + V$$

Left Side

$$\frac{dT}{dt} = \frac{E}{i\hbar} T = \frac{-iE}{\hbar} T$$

Solution

$$T(t) = e^{-iEt/\hbar} = e^{-i\omega t}$$

Definition

$$\frac{E}{\hbar} \equiv \omega$$

Right Side - Time Independent Schrodinger Eqn (TISE)

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + V\phi = E\phi$$

Full solution

$$\psi_E(x,t) = \phi(x)T(t) = \phi_E(x)e^{-iEt/\hbar}$$

- Separated solutions are not necessarily all the solutions.

- Separated solutions are stationary states

$$\rho(x,t) = \Psi^*\psi = \phi_E^* e^{iEt/\hbar}\phi_E e^{-iEt/\hbar}$$

$$= \phi_E^*\phi_E = \rho(x)$$

$\Longrightarrow$ Probability density does not depend on time

$\Longrightarrow \langle x \rangle = $ constant

$\Longrightarrow \langle p \rangle = m\frac{d\langle x\rangle}{dt} = 0$

$$\Rightarrow \langle Q(x,p)\rangle = \int \phi_E^* Q(x,p)\, \phi_E \, dx$$

$$= \text{constant}$$

$\Rightarrow$ All averages of a stationary state are constant.

<u>What is E?</u> We want the operator $Q(x,p)$ that represents the total energy of the system classically.

$$E = \frac{1}{2} m v^2 + V(x)$$

If the total energy is written in terms of the position and momentum, we call it the <u>Hamiltonian</u> of the system.

$$E = H(x,p) = \frac{p^2}{2m} + V$$

Calculate the average total energy of a stationary state.

$$\langle H \rangle = \int \psi_E^* \, H(x, \hat{p}) \, \psi_E \, dx$$

$$= \int \psi_E^* \left(\frac{\hat{p}^2}{2m} + V \right) \psi_E \, dx \qquad \hat{p} = \frac{\hbar}{i} \frac{d}{dx}$$

$$= \int \phi_E^* e^{iEt/\hbar} \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V \right) \phi_E \, e^{-iEt/\hbar} \, dx$$

$$= \int \phi_E^* \underbrace{\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V \right) \phi_E}_{E \phi_E} \, dx$$

$$= \int \phi_E^* \, E \, \phi_E \, dx = E \int \phi_E^* \phi_E \, dx$$

$$= E \implies$$ The separation constant is the average total energy of the state.

Calculate the uncertainty in the energy of ψ_E.

$$\sigma_E = \sqrt{\langle H^2 \rangle - \langle H \rangle^2}$$

$$\langle H^2 \rangle = \int \psi_E^* \left(\frac{\hat{p}^2}{2m} + V \right)^2 \psi_E \, dx$$

$$= \int \psi_E^* E^2 \psi_E \, dx = E^2$$

$$\sigma_E = \sqrt{E^2 - E^2} = 0$$

$\Longrightarrow$ The stationary state $\psi_E = \phi_E e^{-iEt/\hbar}$ is a state of definite energy E.

Properties

(1) TISE is linear, so if ψ_{E_1} and ψ_{E_2} are solutions then

$$\psi = \alpha \psi_{E_1} + \beta \psi_{E_2}$$

is also a solution.

(2) Completeness Any solution to the SE can be constructed from stationary states.

$$\psi(x,t) = \sum c_n \phi_{E_n} e^{-iE_n t/\hbar}$$

or

$$\psi(x,t) = \int c(E)\, \phi(E,x)\, e^{-iEt/\hbar}\, dE$$

Free Particle

Suppose $V = 0$, the TISE becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\phi_E}{dx^2} = E\phi_E$$

Solution

$$\phi_E = e^{\pm i\sqrt{\frac{2mE}{\hbar^2}}\,x}$$

Definition

$$k = \pm\sqrt{\frac{2mE}{\hbar^2}}$$

$$E = \frac{\hbar^2 k^2}{2m} \qquad p = \hbar k$$

$$= \hbar\omega$$

Right traveling $\quad \psi_R(x,t) = e^{ikx - i\omega t}$

Left Traveling $\quad \psi_L(x,t) = e^{-ikx - i\omega t}$

$\Longrightarrow$ Note, free particle states are not normalizable.

$\Longrightarrow$ Momentum

$$p\,\psi_E = \hat{p}\, e^{ikx-i\omega t} = \frac{\hbar}{i}\frac{d}{dx}\, e^{ikx-i\omega t}$$

$$= \hbar k\, \psi_E$$

$\Longrightarrow$ Eigenstate of Momentum

$\Longrightarrow$ Probability Current J

$$J = \frac{i\hbar}{2m}\Big(\underbrace{\psi_R \frac{\partial \psi_R^*}{\partial x}}_{-ik} - \underbrace{\psi_R^* \frac{\partial \psi_R}{\partial x}}_{ik}\Big)$$

$$= \frac{i\hbar}{2m}\left(-2ik\right)$$

$$= \frac{\hbar k}{m}$$ $\Longrightarrow$ constant probability current

$J =$ density $\cdot$ velocity

$\quad\quad\quad\quad 1 \quad\quad\quad\quad p/m$

Free Particle Momentum Space Wave Function

If $\quad \phi(x) = \dfrac{e^{ip'x/\hbar}}{\sqrt{2\pi\hbar}} \quad$ what is $\bar{\phi}(p)$

$$\bar{\phi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \phi(x)\, e^{-ipx/\hbar}\, dx$$

$$= \frac{1}{2\pi\hbar} \int e^{ip'x/\hbar}\, e^{-px/\hbar}\, dx$$

$$= \frac{1}{2\pi\hbar} \int e^{i(p'-p)x/\hbar}\, dx$$

u - sub $\quad \dfrac{x}{\hbar} = u \quad \dfrac{dx}{\hbar} = du$

$$\bar{\phi}(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(p'-p)u}\, du$$

$$= \delta(p'-p) = \delta(p-p')$$

$\Longrightarrow$ One of the many representations of the delta function.

$$\rho(p) = \bar{\phi}(p)\,\phi(p) = \delta^2(p'-p)$$

$$\int \rho(p)\,dp = \infty$$, but the wave functions were not normalizable anyway.

$\Rightarrow$ Clearly however $p = p'$ with probability 1.