Example

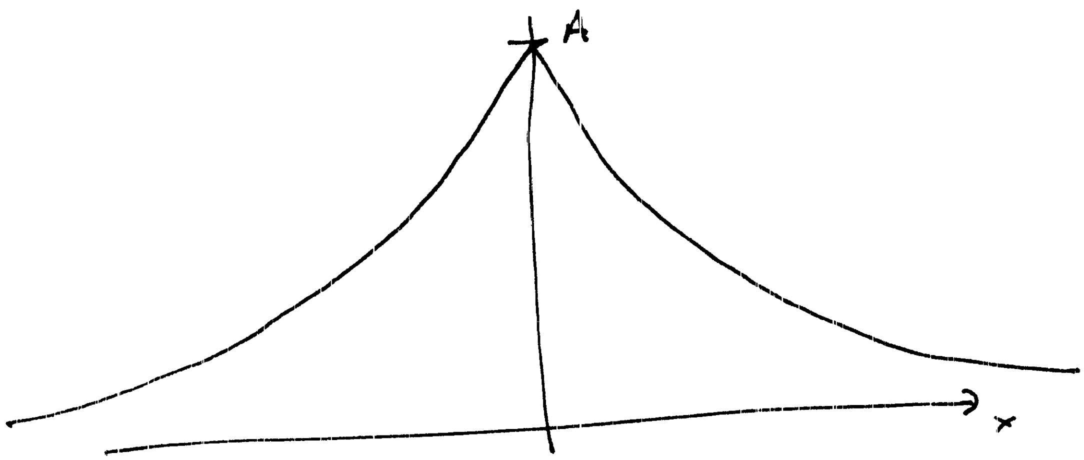
Consider the wave function, $\Psi(x,t) = A e^{-|x|/a} e^{-iEt/\hbar}$

The separate form suggests the wave function is a stationary state of a system with potential V that must be determined.

$\Rightarrow$ Find $\langle x \rangle$, $\langle p \rangle$, $\langle T \rangle$, σ_x, σ_p, V

Normalize

$$1 = AA^* \int_{-\infty}^{\infty} e^{-2|x|/a} \, dx$$



$$\int_{-\infty}^{\infty} e^{-2|x|/a}\, dx = 2\int_0^{\infty} e^{-2x/a}\, dx$$

$$u = 2x/a \qquad du = 2/a\, dx$$

$$2\int_0^{\infty} e^{-2x/a}\, dx = 2\cdot\left(\frac{a}{2}\right)\int_0^{\infty} e^{-u}\, du$$

$$= a\left(-e^{-u}\right)\Big|_0^{\infty} = a$$

$$1 = AA^{*}\cdot a$$

$$A = \frac{1}{\sqrt{a}}$$

$$\psi(x,t) = \phi(x)\, e^{-iEt/\hbar}$$

$$\phi(x) = \frac{1}{\sqrt{a}}\, e^{-|x|/a}$$

Average Position

$$\langle x \rangle = \int \psi^* x \psi \, dx = \int_{-\infty}^{\infty} x \phi^*(x) \phi(x) \, dx$$

$$= A^2 \int_{-\infty}^{\infty} x \, e^{-2|x|/a} \, dx = 0$$

odd function - even range

Uncertainty in Position

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi^* x^2 \psi \, dx$$

$$= A^2 \int_{-\infty}^{\infty} x^2 e^{-2|x|/a} \, dx$$

$$= \frac{2}{a} \int_0^{\infty} x^2 e^{-2x/a} \, dx$$

$$= \left(\frac{2}{a}\right) \frac{a^3}{4} = \frac{a^2}{2}$$ Wolfram

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

$$= \sqrt{\frac{a^2}{2} - 0} = \frac{a}{\sqrt{2}}$$

At this point, we can realize the phase factor $e^{-iEt/\hbar}$ is going to cancel out of each expectation value,

$$\langle Q\rangle = \int \psi^* Q \psi \, dx = \int \phi^* Q \phi \, dx$$

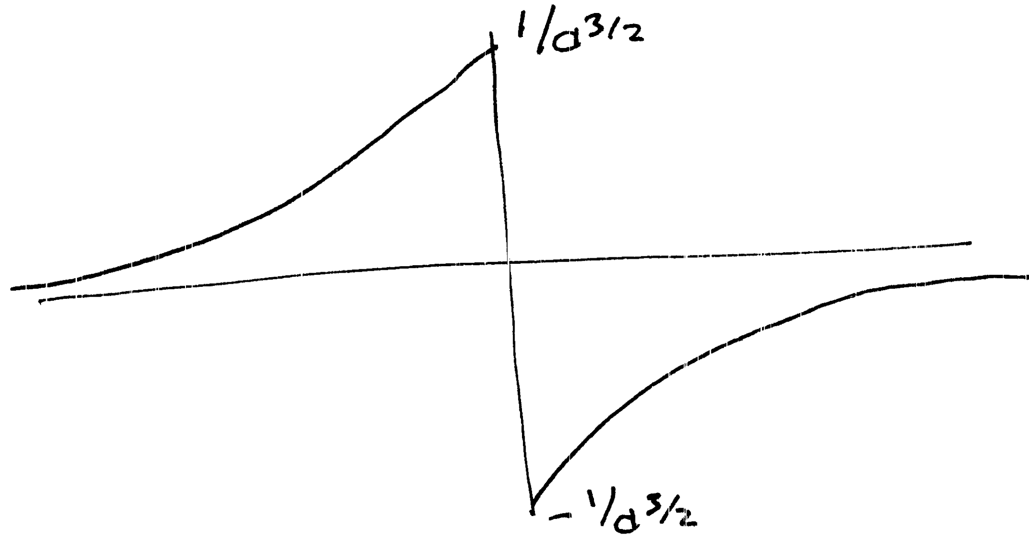
where $\psi(x,t) = \phi(x)\, e^{-iEt/\hbar}$

$$\phi(x) = \frac{1}{\sqrt{a}} e^{-|x|/a}$$

We will need the derivatives of $\phi(x)$

$$\phi(x) = \begin{cases} \frac{1}{\sqrt{a}} e^{+x/a} & x < 0 \\ \frac{1}{\sqrt{a}} e^{-x/a} & x > 0 \end{cases}$$

$$\frac{d\phi}{dx} = \begin{cases} \frac{1}{a^{3/2}} e^{x/a} & x < 0 \\ -\frac{1}{a^{3/2}} e^{-x/a} & x > 0 \end{cases}$$



The function undergoes a step of height $-\frac{2}{a^{3/2}}$ at $x = 0$. The second derivatives,

$$\frac{d^2\phi}{dx^2} = \begin{cases} \frac{1}{a^{5/2}} e^{+x/a} & x < 0 \\ -\frac{2}{a^{3/2}} \delta(x) & x = 0 \\ \frac{1}{a^{5/2}} e^{-x/a} & x > 0 \end{cases}$$

The delta function $\delta(x)$

Consider a step function $\Theta(x)$

$$\Theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

We can write the step function as the integral of the Dirac delta function

$$\Theta(x) = \int_{-\infty}^{x} \delta(x)\, dx$$

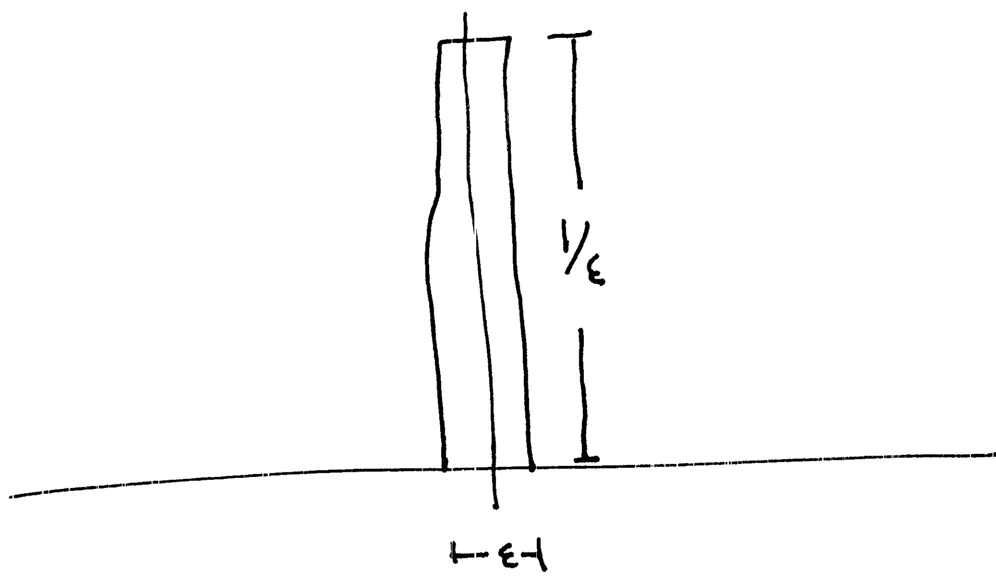
By the fundamental thm of calculus,

$$\frac{d\Theta(x)}{dx} = \delta(x)$$

$\Rightarrow$ The derivative of a step function is a delta function.

Imagine the delta function as the limit of a very narrow function that always has area 1

$$\delta_\varepsilon = \begin{cases} 0 & x < -\varepsilon/2 \\ 1/\varepsilon & -\varepsilon/2 < x < \varepsilon/2 \\ 0 & x > \varepsilon/2 \end{cases}$$



$$\int_{-\infty}^{-\varepsilon/2} \delta_\varepsilon(x)\,dx = 0$$

$$\int_{-\infty}^{\varepsilon/2} \delta_\varepsilon(x)\,dx = 1$$

If we let $\varepsilon \to 0$, we get the step function

Delta Functions make integration easy

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

or

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

Back to our calculation

Average Momentum $\qquad p = \frac{\hbar}{i}\frac{d}{dx}$

$$\langle p \rangle = \int \phi^* \left(\frac{\hbar}{i}\frac{d}{dx}\right) \phi \, dx$$

$$= \frac{\hbar}{i}\int_{-\infty}^{0} \frac{1}{a^2} e^{+2x/a}\,dx + \frac{\hbar}{i}\int_{0}^{\infty} \frac{-1}{a^2} e^{-2x/a}\,dx$$

sign change

$$= 0$$

We could also deduce this from the fact that $\langle x \rangle$ is constant $\quad \langle p \rangle = m\frac{d\langle x \rangle}{dt}$

$$\langle p^2 \rangle = \int \phi^* \left(\frac{\hbar}{i} \frac{d}{dx} \right)^2 \phi \, dx$$

$$= -\hbar^2 \int \phi^* \frac{d^2\phi}{dx^2} \, dx$$

$$= -\hbar^2 \int_{-\infty}^{0} \frac{1}{\sqrt{a}} e^{x/a} \cdot \frac{1}{a^{5/2}} e^{x/a} \, dx$$

$$-\hbar^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} e^{-|x|/a} \delta(x) \cdot \left(\frac{-2}{a^{3/2}} \right) dx$$

$$-\hbar^2 \int_{0}^{\infty} \frac{1}{\sqrt{a}} e^{-x/a} \cdot \frac{1}{a^{5/2}} e^{-x/a} \, dx$$

$$= -\frac{2\hbar^2}{a^3} \int_0^\infty e^{-2x/a} \, dx$$

$$+ \frac{2\hbar^2}{a^2} \underbrace{\int_{-\infty}^{\infty} \delta(x) e^{-|x|/a} \, dx}_{e^{-0/a}}$$

$$= \frac{2\hbar^2}{a^2} - \frac{2\hbar^2}{a^3} \underbrace{\int_0^\infty e^{-2x/a}}_{a/2}$$

$$\langle p^2 \rangle = \frac{2\hbar^2}{a^2} - \frac{2\hbar^2}{a^3} \cdot \frac{a}{2}$$

$$= \frac{\hbar^2}{a^2}$$

Uncertainty in Momentum

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

$$= \sqrt{\frac{\hbar^2}{a^2} - 0} = \frac{\hbar}{a}$$

Uncertainty Relation

$$\sigma_x \sigma_p = \frac{a}{\sqrt{2}} \cdot \frac{\hbar}{a} = \frac{\hbar}{\sqrt{2}} > \frac{\hbar}{2}$$

What about V?

$$p^2 \phi = -\hbar^2 \frac{d^2\phi}{dx^2}$$

$$= \frac{-\hbar^2}{a^2}\phi(x) + \frac{2\hbar^2}{a}\sigma(x)\phi(x)$$

$$\frac{p^2}{2m}\phi = -\frac{\hbar^2}{2ma^2}\phi + \frac{2\hbar^2}{2ma}\sigma(x)\phi(x)$$

$$\frac{-\hbar^2}{2m}\frac{d^2\phi}{dx^2} = E\phi - V\phi$$

$$\Longrightarrow \quad E = \frac{-\hbar^2}{2ma^2}$$

$$V(x) = -\frac{\hbar^2}{ma}\sigma(x)$$

Note, dimensions check

$$E = \frac{\hbar^2 k^2}{2m} \qquad [k] = \frac{1}{\ell}$$

$$[\sigma(x)] = \frac{1}{\ell}$$

Momentum Probability Density

$$\Psi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \psi(x,t)\, dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\, e^{-i\omega t} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \frac{1}{\sqrt{a}}\, e^{-|x|/a}\, dx$$

$$= \frac{1}{\sqrt{2\pi\hbar a}}\, e^{-i\omega t} \left[\int_{-\infty}^{0} \exp\left[\left(\frac{1}{a} - \frac{ip}{\hbar}\right)x\right] dx \right.$$

$$\left. + \int_{0}^{\infty} \exp\left[\left(-\frac{1}{a} - \frac{ip}{\hbar}\right)x\right] dx \right]$$

$$\int_0^{\infty} e^{-bx}\, dx = -\frac{1}{b}\, e^{-bx}\Big|_0^{\infty} = \frac{1}{b}$$

$$\int_{-\infty}^{0} e^{bx}\, dx = \frac{1}{b} e^{bx}\Big|_{-\infty}^{0} = \frac{1}{b}$$

$$[\;] = \frac{1}{\frac{1}{a} - \frac{ip}{\hbar}} + \frac{1}{\frac{1}{a} + \frac{ip}{\hbar}} = \frac{\frac{1}{a} + \frac{ip}{\hbar} + \frac{1}{a} - \frac{ip}{\hbar}}{\frac{1}{a^2} + \frac{p^2}{\hbar^2}}$$

$$[\,] = \frac{2}{a} \cdot \frac{1}{\frac{1}{a^2} + \frac{p^2}{\hbar^2}}$$

$$= \frac{2a}{1 + \frac{p^2 a^2}{\hbar^2}}$$

$$\overline{\Psi}(p,t) = \frac{1}{\sqrt{2\pi\hbar a}} e^{-i\omega t} \cdot \frac{2a}{1 + \frac{p^2 a^2}{\hbar^2}}$$

$$= \sqrt{\frac{2a}{\pi\hbar}}\, e^{-i\omega t} \cdot \frac{1}{1 + p^2 a^2/\hbar^2}$$

<u>Check Dimensions</u>

$$[\overline{\Psi}(p,t)] = \frac{1}{\sqrt{p}} \; *$$

$$\left[\frac{a}{\hbar}\right] = \left[\frac{1}{\hbar k}\right] = \frac{1}{p} \quad \checkmark$$