Uncertainty Principle

The general uncertainty principle for two quantities a and b represented by observables $\hat{A}$ and $\hat{B}$ is

$$\sigma_a^2 \sigma_b^2 \geq \left(\frac{1}{2i} \langle \psi | [\hat{A}, \hat{B}] | \psi \rangle \right)^2$$

$\Rightarrow$ If two operators commute, the physical quantities represented by the operators can mutually be measured with infinite precision.

Ex Hegenberg Uncertainty Principle

$$\sigma_x^2 \sigma_p \geq \left(\frac{1}{2i} \langle \psi | \underbrace{[\hat{x}, \hat{p}]}_{i\hbar} | \psi \rangle \right)^2$$

$$\geq \left(\frac{i\hbar}{2i} \right)^2$$

$$\geq \left(\frac{\hbar}{2} \right)^2$$

Proof

Define operators that measure difference from mean

$$\Delta\hat{A} = \hat{A} - \langle a \rangle$$

$$\Delta\hat{B} = \hat{B} - \langle b \rangle$$

$$\sigma_a^2 = \langle \psi | (\Delta\hat{A})^2 | \psi \rangle \qquad \sigma_b^2 = \langle \psi | \Delta\hat{B}^2 | \psi \rangle$$

Define

$$|f\rangle = \Delta\hat{A}|\psi\rangle \qquad |g\rangle = \Delta\hat{B}|\psi\rangle$$

$$\langle f| = \langle \psi | \Delta\hat{A} \qquad \langle g| = \langle \psi | \Delta\hat{B}$$

since $\Delta\hat{A}$, $\Delta\hat{B}$ Hermitian.

$$\sigma_a^2 = \langle f | f \rangle \qquad \sigma_b^2 = \langle g | g \rangle$$

Schwartz Inequality - For any two vectors

$$\langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2$$

$$= \langle f | g \rangle^* \langle f | g \rangle$$

For any complex number,

$$|z|^2 = z^* z = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 \geq \mathrm{Im}(z)^2$$

$$\mathrm{Im}(z) = \frac{z - z^*}{2i}$$

therefore

$$|\langle f|g\rangle|^2 \geq \left(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i} \right)^2$$

<u>Work on $\langle f|g\rangle$</u>

$$\langle f|g\rangle = \langle \psi|(\hat{A} - \langle a\rangle)(\hat{B} - \langle b\rangle)|\psi\rangle$$

$$= \langle \psi|\hat{A}\hat{B}|\psi\rangle + \langle a\rangle\langle b\rangle\langle \psi|\psi\rangle$$

$$- \langle b\rangle \underbrace{\langle \psi|\hat{A}|\psi\rangle}_{\langle a\rangle} - \langle a\rangle \underbrace{\langle \psi|\hat{B}|\psi\rangle}_{\langle b\rangle}$$

$$\langle f|g\rangle = \langle \psi|\hat{A}\hat{B}|\psi\rangle - \langle a\rangle\langle b\rangle$$

$$\langle g|f\rangle = \langle f|g\rangle^{+} = \langle\psi|\hat{B}\hat{A}|\psi\rangle - \langle a\rangle\langle b\rangle$$

$$\langle f|g\rangle - \langle g|f\rangle = \langle\psi|\hat{A}\hat{B}|\psi\rangle - \langle\psi|\hat{B}\hat{A}|\psi\rangle$$

$$= \langle\psi|[\hat{A},\hat{B}]|\psi\rangle$$

So $\sigma_a^2\sigma_b^2 = \langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2$

$$\geq \left(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i}\right)^2$$

$$= \left(\frac{1}{2i}\langle\psi|[\hat{A},\hat{B}]|\psi\rangle\right)^2$$