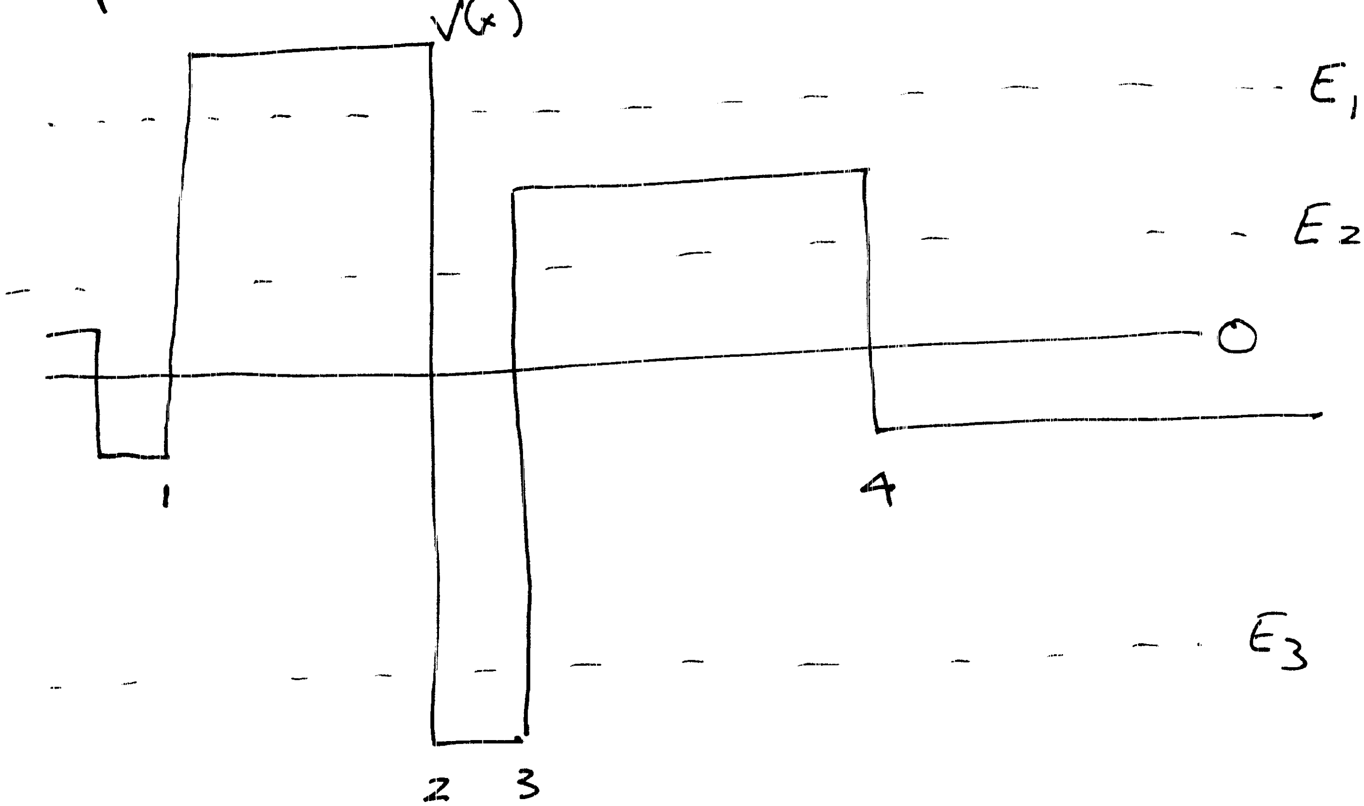
Particles and Waves II

We are going to consider particles trapped in potentials that look like this



Classically, the behavior of the particle depends on the relation of the total energy of the particle E to the potential energy $V(x)$. Regions where $E < V(x)$ are forbidden. Therefore, a particle with energy E_3 would bounce back and forth in the well, between points (2) and (3).

A particle with energy E_2 ~~would be~~ if incident from the left would be reflected at point ① 100% of the time. A particle with energy E_2 could also be trapped forever between ② and ③. If incident from the right, the particle would reflect at point ④. A particle with energy E_2 would NEVER escape the potential well between ② and ③ if placed there.

Since V is constant at most points, we can solve the TISE at points where the potential isn't changing.

TISE

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + V\phi = E\phi$$

$$\frac{d^2\phi}{dx^2} + \frac{2m(E-V)}{\hbar^2}\phi = 0$$

$$\Downarrow$$

$$\frac{d^2\phi}{dx^2} + k^2\phi = 0$$

Which we already know how to solve

$$\phi = e^{\pm ikx}$$

$$k = \sqrt{\frac{2m(E-V)}{\hbar^2}}$$

Looks very much like the free particle, but be careful, $E \neq \hbar^2 k^2/2m$, so the time dependence <u>CANNOT</u> be written $e^{\pm \frac{i\hbar k^2}{2m} t}$.

<u>A few properties</u>

- The effect of the potential was to lower the velocity if $V>0$ and increase the velocity if $V<0$

$$V_{phase} = \frac{\omega}{k} = \frac{E}{\hbar k}$$

- The TISE has a solution in the classically forbidden region, $E<V$

If $E < V$, $E - V < 0$

$$k = \sqrt{\frac{2m(E-V)}{\hbar^2}} = \sqrt{(-1)\frac{2m(V-E)}{\hbar^2}}$$

$$= iK \qquad K = \sqrt{\frac{2m(V-E)}{\hbar^2}}$$

real

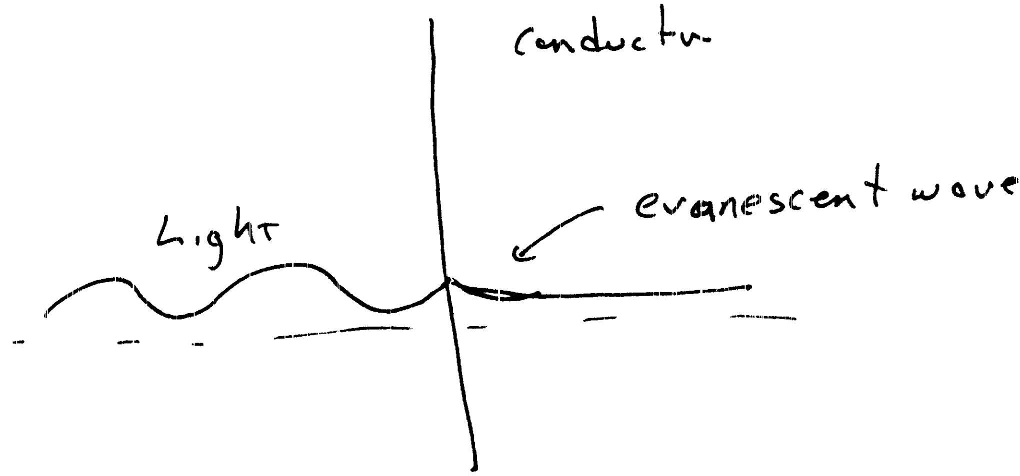
If $E < V$,

$$\phi = e^{\pm ikx} = e^{\pm i(iK)x} = e^{\mp Kx}$$

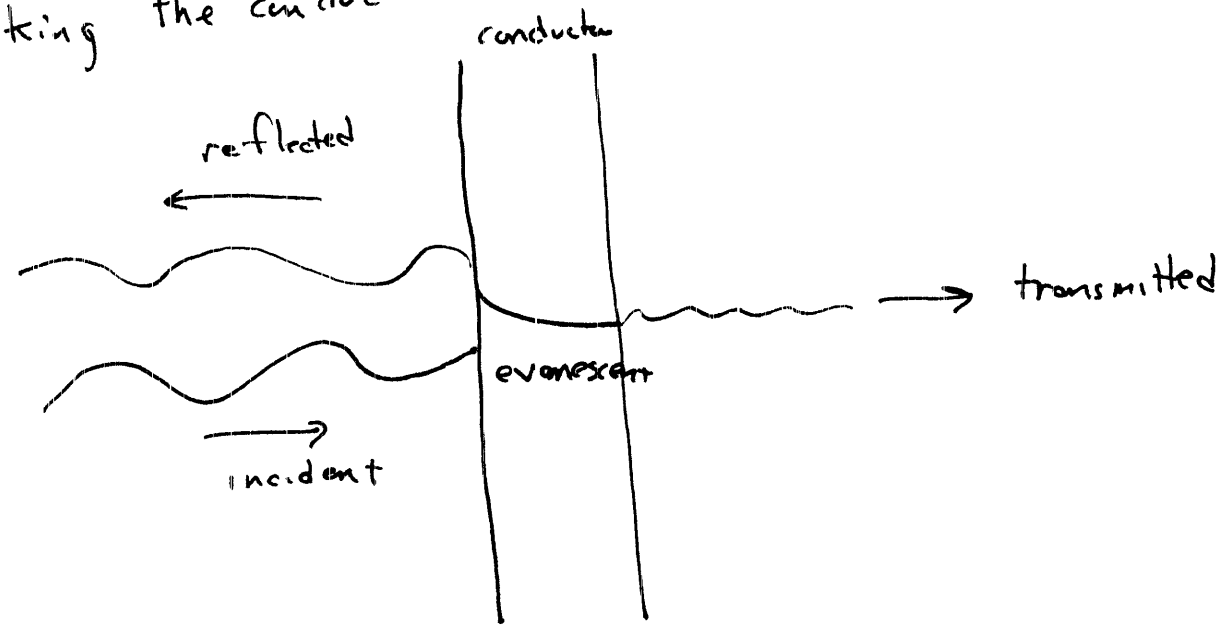
$\Longrightarrow$ The wave decays exponentially is the classically forbidden regions.

If we're taking on the whole particles behave as waves thing this is no suprise.
Most waves have an exponentially decaying part, called an evanescent wave, when the wave encounters a forbidden region.

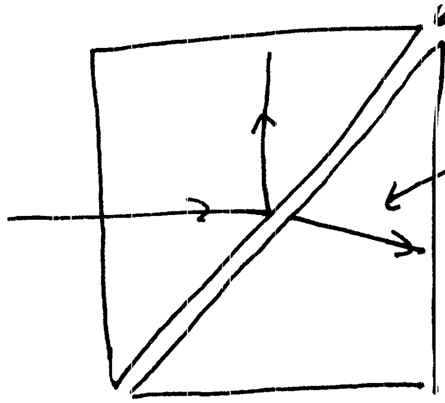
Light is completely reflected from conductor



You can even see the evanescent wave if you provide a medium where the wave can travel before its amplitude is zero, for example by making the conductor thin



This effect is easier to see with two prisms and a incident angle greater than the critical angle.



<u>Boundary Conditions</u> - Boundary conditions come from the physics. The only physics we have is the SE.

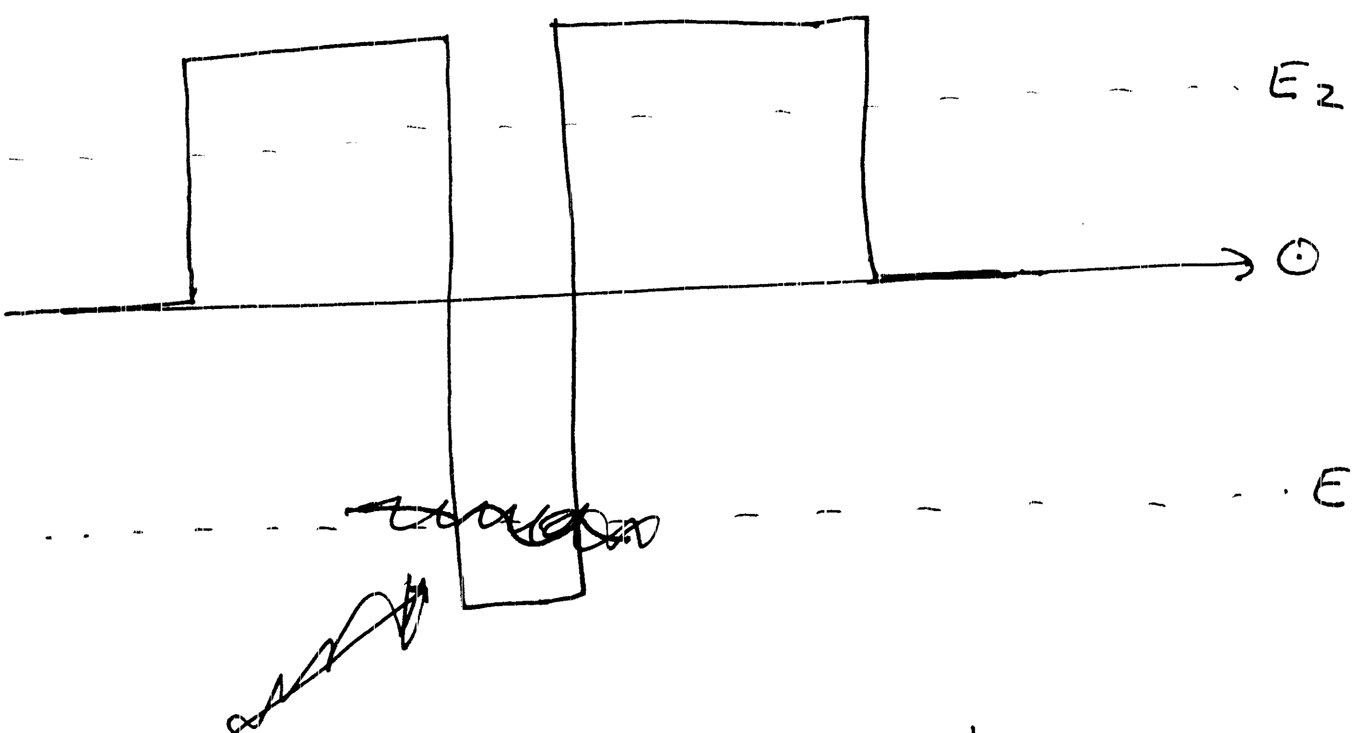
① $\psi(x,t)$ and therefore $\phi(x)$ is <u>continuous</u>

[If not $-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} \propto \delta'(x)$ at the discontinuity. This implies a potential $V(x) \propto \delta'(x)$ and no such potential exists.]

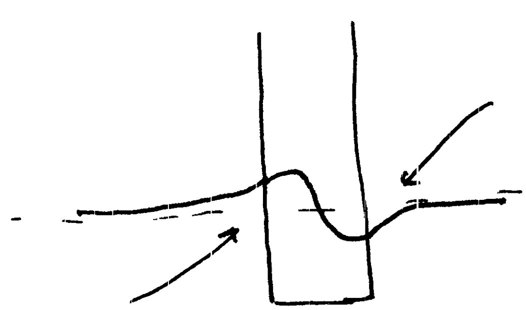
2) $\frac{d\phi}{dx}$ is continuous except if

$$V(x) = V_0 \sigma(x)$$

What does all this imply about a particle, trapped in a potential like we considered earlier?



Wave function for bound particle



$\phi^*\phi$ not zero outside of well

$\Longrightarrow$ We can find particle outside of the well.

Wave function for unbound particle