Section - Rigid Bodies

<u>Rigid Body</u> is a system of particles whose <u>relative</u> positions is fixed.

<u>Center of Mass</u> $\vec{r}_{cm} = (x_{cm}, y_{cm}, z_{cm})$

$$x_{cm} = \frac{1}{M}\sum_i x_i m_i$$

or

$$x_{cm} = \frac{1}{M}\int x \, dm$$

where $dm = \rho \, dV$ or σdA or λdl

where $\rho \equiv$ mass/volume

$\sigma \equiv$ mass/area

$\lambda \equiv$ mass/length.

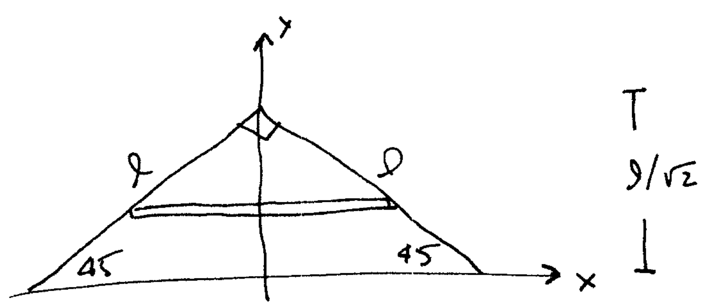
Center of Mass of Composite Body — Suppose a body is made of separate parts with known centers of mass m_1, $\vec{r}_1^{\,cm}$, m_2, $\vec{r}_2^{\,cm}$,

then $$\vec{r}_{cm} = \frac{1}{M} \sum m_i \vec{r}_i^{\,cm}$$

$\Rightarrow$ Evident from definition of CM.

Symmetry — If a body has a line or plane of symmetry, the center of mass lies on that line or plane.

Example Center of Mass of isoceles right triangular lamina, mass density σ.



Total mass $M = \frac{1}{2}$ base $\times$ height $= \frac{1}{2}\ell^2\sigma$

By symmetry the center of mass lies on the y-axis, $x_{cm} = 0$

$$y_{cm} = \frac{1}{M}\int y\,dm$$

The length of the strip drawn is $2(\ell/\sqrt{2} - y)$

$$dm = (\sigma)(2)(\ell/\sqrt{2} - y)\,dy$$

$$y_{cm} = \frac{1}{M}\int y\,dm = \frac{2\sigma}{M}\int_0^{\ell/\sqrt{2}} y(\ell/\sqrt{2} - y)\,dy$$

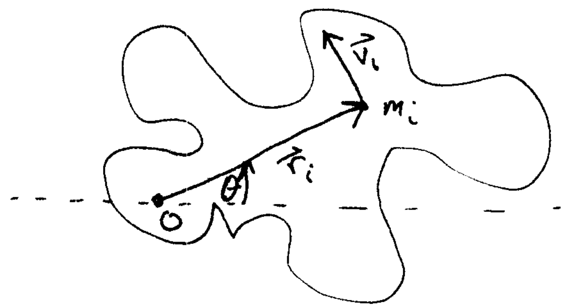
$$= \frac{2\sigma}{M}\left[\frac{y^2\ell}{2\sqrt{2}} - \frac{y^3}{3}\right]_0^{\ell/\sqrt{2}}$$

$$= \frac{2\sigma}{M}\left[\frac{\ell^3}{4\sqrt{2}} - \frac{\ell^3}{6\sqrt{2}}\right]$$

$$= \frac{2\sigma\ell^3}{M}\left[\frac{1}{12\sqrt{2}}\right]$$

$$y_{cm} = \frac{\ell}{3\sqrt{2}}$$

Section - Rotation about Fixed Axis



If the body is rigid and is constrained to rotate about fixed axis O (z axis). All the particles move in circles with angular velocity ω and

speed $v_i = r_i \omega$

$$\vec{r}_i = r_i(\cos\theta, \sin\theta, 0)$$

$$\vec{v}_i = r_i(-\sin\theta\,\dot{\theta}, \cos\theta\,\dot{\theta}, 0)$$

$$= \omega r_i(-\sin\theta, \cos\theta, 0)$$

Observation gives, where $\vec{\omega} = \omega\hat{k}$

$$\boxed{\vec{v}_i = \vec{\omega} \times \vec{r}_i}$$

Kinetic Energy (Pure Rotation)

$$T_{rot} = \frac{1}{2} \sum_i m_i v_i^2 = \frac{\omega^2}{2} \sum_i m_i r_i^2$$

Define <u>Moment of Inertia</u> about axis of rotation z

$$I_z = \frac{1}{2} \sum_i m_i r_i^2 = \frac{1}{2} \sum_i m_i (x_i^2 + y_i^2)$$

<u>Angular Momentum</u>

$$\vec{L} = \sum \vec{L}_i = \sum_i m_i \vec{r}_i \times \vec{v}_i$$

$$\vec{L}_i^{z} = m_i \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & 0 \\ \dot{x}_i & \dot{y}_i & 0 \end{vmatrix}$$

$$= m_i (x_i \dot{y}_i - y_i \dot{x}_i) \hat{k}$$

$$\vec{v}_i = r_i(-\sin\theta, \cos\theta)\omega$$

$$\dot{x}_i = -r_i \omega \sin\theta \qquad \dot{y}_i = r_i \omega \cos\theta$$

$$\vec{L}_i = m_i \omega \left(x_i r_i \cos\theta + y_i r_i \sin\theta \right)$$

$$= m_i \omega \left(x_i^2 + y_i^2 \right)$$

$$\vec{L} = \omega \sum m_i \left(x_i^2 + y_i^2 \right) = I_z \omega$$

<u>Torque</u> $\quad \dfrac{dN_z}{dt} = \dfrac{d\, I_z \omega}{dt} = I_z \dot{\omega}$

Lecture 3/28/2003

Section - Comparison of Rocket Methods

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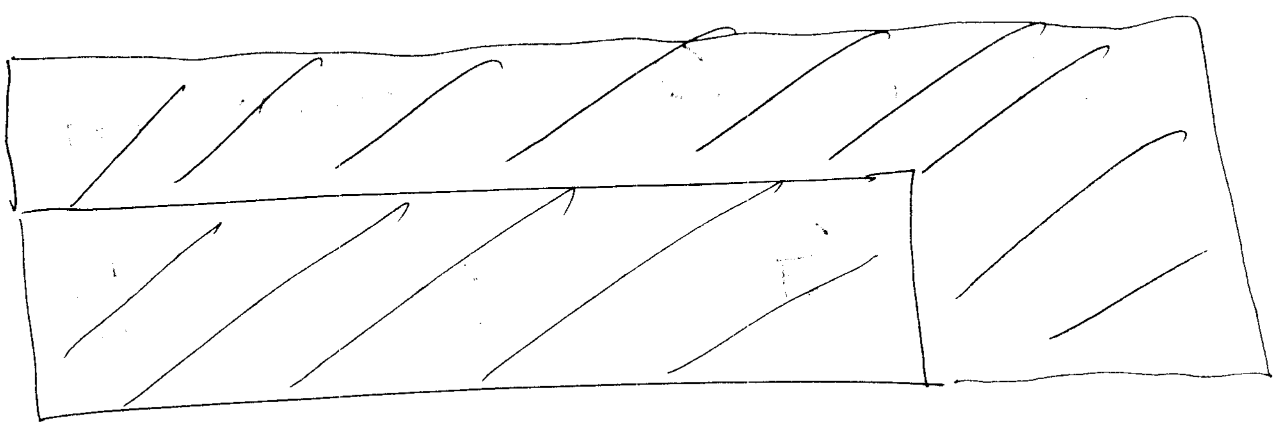
Let mass per unit length be λ

Lifted Mass λm

Total Mass λL



$\boxed{\text{Force of gravity and normal force cancel for rope lying on table.}}$

$= M a_{cm}$

M

$-V$

* Adding Mass

* Added mass moves with negative velocity

Section - Rigid Bodies

Rigid Body - A system of particles whose relative positions are fixed. This means in the coordinate system where the body is stationary, $\vec{r}_i - \vec{r}_j$ does not depend on time.

Center of Mass $\vec{r}_{cm} = (x_{cm}, y_{cm}, z_{cm})$

$$x_{cm} = \frac{1}{M} \sum_i x_i m_i$$

- or -

$$x_{cm} = \frac{1}{M} \int x \, dm$$

dm is an element of mass

$$dm = \underline{\rho \, dV} \quad \text{or} \quad \underline{\sigma \, dA} \quad \text{or} \quad \underline{\lambda \, d\ell}$$

where ρ is the volume mass density
σ is the surface mass density
λ is the linear mass density

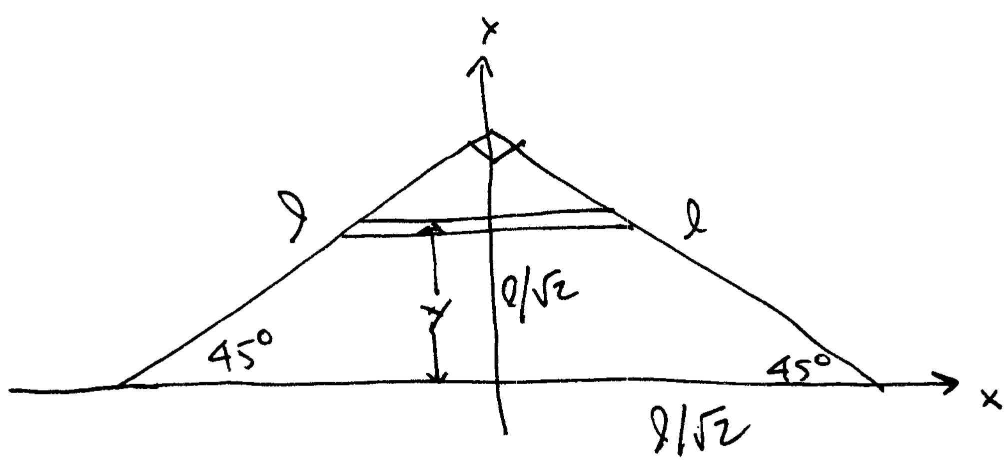
Center of Mass of Composite Body - Suppose a body is made of N parts whose masses are $m_1 \vec{r}_1^{\,cm}$ and $m_2 \vec{r}_2^{\,cm}$ .....

Then the Center of Mass of the composite body is

$$\vec{r}_{cm} = \frac{1}{m_1 + m_2 \cdots}\left[m_1 \vec{r}_1^{\,cm} + m_2 \vec{r}_2^{\,cm} \cdots\right]$$

Symmetry - If a body has a line or plane of symmetry, the CM lies on that line/plane.

Example - Find CM of isoceles right planar triangle with uniform surface mass density σ.



Total Mass $\quad M = \frac{1}{2}$ base $\cdot$ height $\cdot\ \sigma$

$$= \frac{1}{2}\sigma \ell^2$$

Symmetry $\quad x\cancel{y}$-plane , $\quad yz$ ~~plane~~ (or plane)

$$\cancel{z}_{cm} = 0 \qquad x_{cm} = 0$$

Definition

$$M \cancel{y}_{cm} = \int \cancel{y}\, dm$$

$$dm = (\text{length of strip})\, dy\, \sigma$$

$$= 2\sigma\left(\ell/\sqrt{2} - y\right) dy$$

$$M y_{cm} = 2\sigma \int_0^{\ell/\sqrt{2}} \left(\ell/\sqrt{2} - y\right) y\, dy$$

$$= 2\sigma \left[\frac{\ell}{\sqrt{2}}\frac{y^2}{2} - \frac{y^3}{3}\right]_0^{\ell/\sqrt{2}}$$

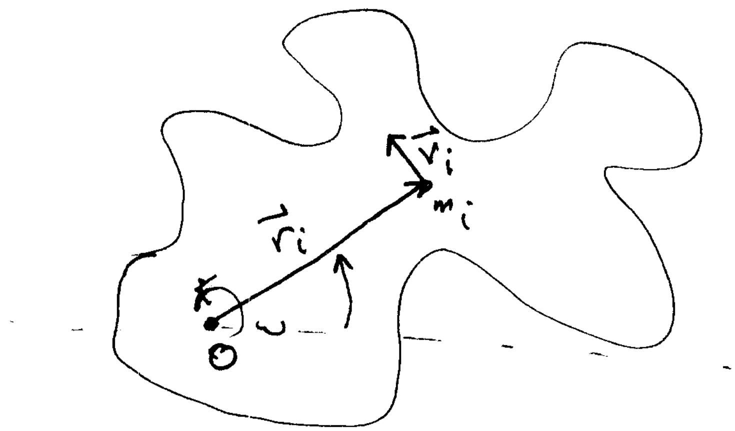
$$= 2\sigma\left[\frac{\ell^3}{4\sqrt{2}} - \frac{\ell^3}{6\sqrt{2}}\right]$$

$$= \frac{\sigma \ell^3}{6\sqrt{2}} = \frac{\sigma \ell^3}{6\sqrt{2}}$$

$$y_{cm} = \frac{\dfrac{\sigma \ell^3}{6\sqrt{2}}}{\frac{1}{2}\sigma \ell^2} = \frac{\ell}{3\sqrt{2}} = \frac{\ell}{3\sqrt{2}}$$

Section - Rotation of Rigid Body about Fixed Axis

Motion of a plane object



If a body is rigid and constrained to rotate about the point O with angular velocity $\vec{\omega}$.

<u>Velocity of any point</u> $\quad v_i = r_i \omega \qquad \omega = \dot{\theta}$

$$\vec{r}_i = r_i(\cos\theta, \sin\theta, 0)$$

$$\vec{v}_i = \dot{\vec{r}}_i = r_i(-\sin\theta\,\dot{\theta}, \cos\theta\,\dot{\theta}, 0)$$

$$= r_i\omega(-\sin\theta, \cos\theta, 0) = r_i\omega\,\hat{e}_\theta$$

<u>This can be written</u>

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

Check Magnitude $\quad |\vec{v}_i| = |\vec{\omega} \times \vec{r}_i| = \omega r_i$

Check Direction $\quad$ Use RHR, direction correct.

Kinetic Energy (Pure Rotation) –

$$T_{rot} = \frac{1}{2}\sum_i m_i \vec{v}_i^{\,2}$$

$$|\vec{v}_i| = \omega r_i$$

$$T_{rot} = \frac{\omega^2}{2}\sum m_i r_i^2 = \frac{1}{2} I \omega^2$$

Moment of Inertia (About O) –

$$I = \sum_i m_i r_i^2 = \sum m_i (x_i^2 + y_i^2)$$

Angular Momentum ($\vec{L}$)

$$\vec{L} = \sum_i \vec{L}_i = \sum_i m_i \vec{r}_i \times \vec{v}_i$$

$$\vec{L}_i = m_i \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & 0 \\ \dot{x}_i & \dot{y}_i & 0 \end{vmatrix}$$

$$= m_i \left(x_i \dot{y}_i - y_i \dot{x}_i \right) \hat{k}$$

$$\dot{\vec{r}}_i = \vec{v}_i = r_i \omega \left(-\sin\theta, \cos\theta, 0 \right)$$

$$\dot{x}_i = -r_i \omega \sin\theta \qquad \dot{y}_i = r_i \omega \cos\theta$$

$$\vec{L}_i = m_i \left(x_i r_i \omega \cos\theta + y_i r_i \omega \sin\theta \right) \hat{k}$$

$$x_i = r_i \cos\theta \qquad y_i = r_i \sin\theta$$

$$\vec{L}_i = m_i \omega \left(x_i^2 + y_i^2 \right) \hat{k}$$

$$\vec{L} = \sum \vec{L}_i = \omega \sum m_i (x_i^2 + y_i^2) \hat{k} = I \omega \hat{k}$$

EOM

$$\vec{N} = \frac{d\vec{L}}{dt}$$

z-component

$$\boxed{N_z = \frac{d\, I\omega}{dt} = I\dot{\omega}}$$

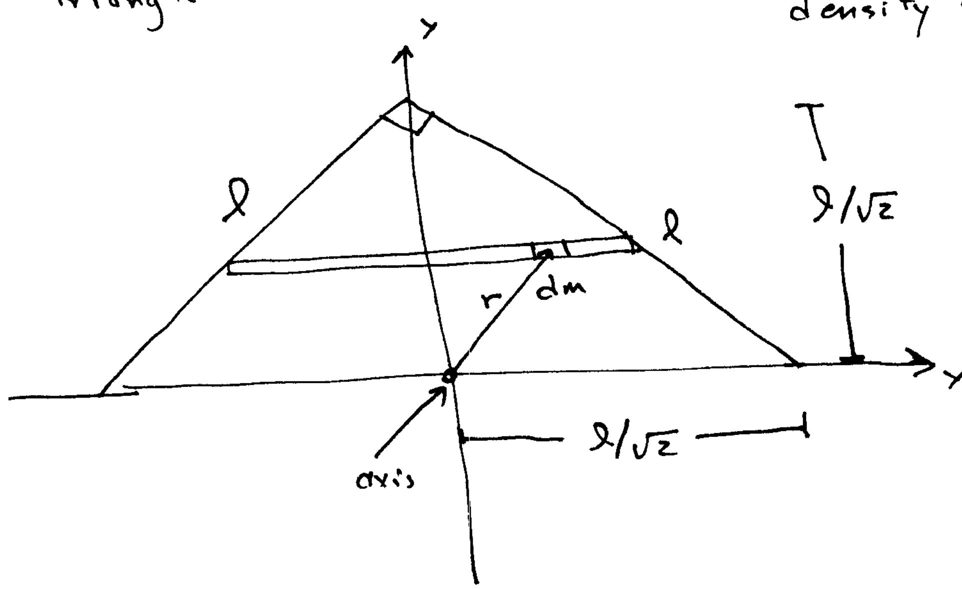
Section - <u>Moment of Inertia</u>

<u>Moment of Inertia</u> - Moment of Inertia about any axis

$$I = \int dm\, r^2$$

where r^2 is the perpendicular distance from the axis.

<u>Example</u> Moment of Inertia of Isoceles Right triangle about z axis. Uniform mass density σ.



<u>Cut tringle into strips</u> -

$$I = \int r^2 \, dm \qquad dm = \sigma \, dx \, dy \qquad r^2 = x^2 + y^2$$

$$= \sigma \int_0^{\ell/\sqrt{2}} dy \int_{-(\ell/\sqrt{2} - y)}^{\ell/\sqrt{2} - y} dx \, (x^2 + y^2)$$

$$= 2\sigma \int_0^{\ell/\sqrt{2}} dy \left[\frac{x^3}{3} + x y^2 \right]_0^{\ell/\sqrt{2} - y}$$

$$= 2\sigma \int_0^{\ell/\sqrt{2}} dy \left[\frac{1}{3} \left(\ell/\sqrt{2} - y \right)^3 + \frac{\ell}{\sqrt{2}} y^2 - y^3 \right]$$

Ask if continue.

$$= \int_0^{\ell/\sqrt{2}} 2\sigma \left[\frac{1}{3} \left(\frac{\ell^3}{2\sqrt{2}} - \frac{3\ell^2}{2} y + \frac{3\ell}{\sqrt{2}} y^2 - y^3 \right] + \frac{\ell}{\sqrt{2}} y^2 - y^3 \right]$$

$$= 2\sigma \int_0^{\ell/\sqrt{2}} \left[\frac{\ell^3}{6\sqrt{2}} - \frac{\ell^2}{2} y + \frac{2\ell}{\sqrt{2}} y^2 - \frac{4}{3} y^3 \right] dy$$

$$= 2\sigma \left(\frac{\ell^3}{6\sqrt{2}} y - \frac{\ell^2 y^2}{4} + \frac{2\ell y^3}{3\sqrt{2}} - \frac{1}{3} y^4 \right)_0^{\ell/\sqrt{2}}$$

Perpendicular Axis Thm - For a thin lamina in the x-y plane.

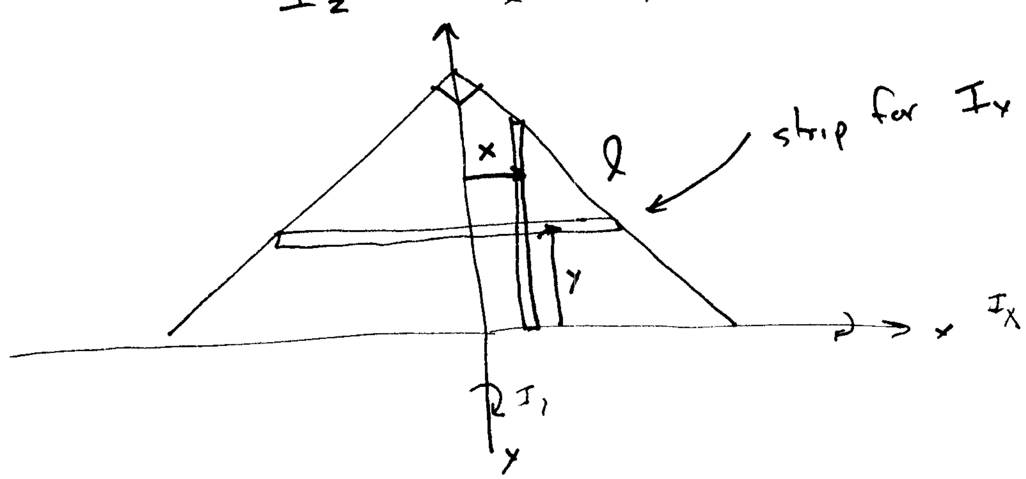
$$I_z = I_x + I_y$$

$$= \underbrace{\sum m_i (x_i^2 + y_i^2)}_{I_z} = \underbrace{\sum m_i x_i^2}_{I_y} + \underbrace{\sum m_i y_i^2}_{I_x}$$

The moment of inertia about an axis perpendicular to the lamina is equal to the sum of the moments of two mutually perpendicular axis through the axis.

Example Back to Right Isoceles Triangle

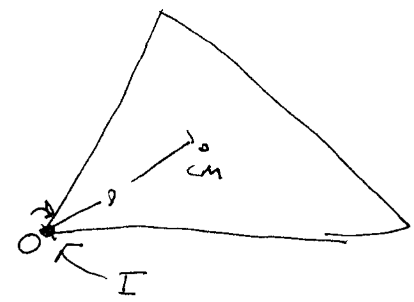
$$I_z = I_x + I_y$$



$$I_x = \int dm\, y^2 = 2\int_0^{l/\sqrt{2}} \sigma (l/\sqrt{2} - y)\, y^2\, dy$$

$$I_y = \int x^2\, dm = \sigma \int_{-\ell/\sqrt{2}}^{\ell/\sqrt{2}} dx\, x^2 \left(\ell/\sqrt{2} - x \right)$$

Parallel Axis Thm



The moment of inertia about an axis O is the sum of the moment of inertia about a parallel axis through the center of mass plus the moment of inertia of the body, treated as a point mass at the center of mass about axis O.

If l is the distance from O to cm,

$$I = I_{cm} + ml^2$$

Radius of Gyration (K) Distance a point mass must be placed from the axis of rotation for the point mass to have the same moment of inertia as the body - Since

$$I_{point} = mk^2,$$

$$K = \sqrt{\frac{I}{m}}$$